Suppose that we are fitting a Gaussian mixture model for data items consisting of a single real value, x, using $K = 2$ components. We have $N = 5$ training cases, in which the values of x are as follows:

$$5, \quad 10, \quad 20, \quad 35, \quad 40$$

We use the EM algorithm to find the maximum likelihood estimates for the model parameters, which are the mixing proportions for the two components, π_1 and π_2, and the means for the two components, μ_1 and μ_2. The standard deviations for the two components are fixed at 10.

Suppose that at some point in the EM algorithm, the E step found that the responsibilities $\gamma(z_{nk})$ for sample n and mixture k are as follows:

	Mixture 1	Mixture 2
Sample 1	0.2	0.8
Sample 2	0.2	0.8
Sample 3	0.9	0.1
Sample 4	0.9	0.1
Sample 5	0.9	0.1

(a) What are the values of parameters π_1, π_2, μ_1, and μ_2 to be found in the next M step of the algorithm?

$$\mu_1 = \frac{\sum_n \gamma(z_{n1}) x_n}{\sum_n \gamma(z_{n1})} = \frac{0.2\times5+0.2\times10+0.9\times20+0.9\times35+0.9\times40}{0.2+0.2+0.9+0.9+0.9} = 28.54$$

$$\mu_2 = \frac{\sum_n \gamma(z_{n2}) x_n}{\sum_n \gamma(z_{n2})} = \frac{0.8\times5+0.8\times10+0.1\times20+0.1\times35+0.1\times40}{0.2+0.2+0.9+0.9+0.9} = 11.32$$

$$\pi_1 = \frac{\sum_n \gamma(z_{n1})}{\sum_k \sum_n \gamma(z_{nk})} = \frac{0.2+0.2+0.9+0.9+0.9}{5} = 0.62$$

$$\pi_2 = \frac{\sum_n \gamma(z_{n2})}{\sum_k \sum_n \gamma(z_{nk})} = \frac{0.8+0.8+0.1+0.1+0.1}{5} = 0.38$$

(b) What are the new responsibilities $\gamma(z_{nk})$?

The new responsibilities for each sample belonging to Mixture 1 are:

$$\gamma(z_{11}) \propto \pi_1 \mathcal{N}(x_1|\mu_1, 10) = 0.62 \times \mathcal{N}(5|28.54, 10) = 6.973 \times 10^{-14}$$

$$\gamma(z_{21}) \propto \pi_1 \mathcal{N}(x_2|\mu_1, 10) = 0.62 \times \mathcal{N}(10|28.54, 10) = 2.623 \times 10^{-14}$$

$$\gamma(z_{31}) \propto \pi_1 \mathcal{N}(x_3|\mu_1, 10) = 0.62 \times \mathcal{N}(20|28.54, 10) = 0.00203$$

$$\gamma(z_{41}) \propto \pi_1 \mathcal{N}(x_4|\mu_1, 10) = 0.62 \times \mathcal{N}(35|28.54, 10) = 0.00968$$

$$\gamma(z_{51}) \propto \pi_1 \mathcal{N}(x_5|\mu_1, 10) = 0.62 \times \mathcal{N}(40|28.54, 10) = 0.000109$$

The new responsibilities for each sample belonging to Mixture 2 are:

$$\gamma(z_{12}) \propto \pi_2 \mathcal{N}(x_1|\mu_2, 10) = 0.38 \times \mathcal{N}(5|11.32, 10) = 0.0162$$

$$\gamma(z_{22}) \propto \pi_2 \mathcal{N}(x_2|\mu_2, 10) = 0.38 \times \mathcal{N}(10|11.32, 10) = 0.0004501$$

$$\gamma(z_{32}) \propto \pi_2 \mathcal{N}(x_3|\mu_2, 10) = 0.38 \times \mathcal{N}(20|11.32, 10) = 1.9012 \times 10^{-10}$$

$$\gamma(z_{42}) \propto \pi_2 \mathcal{N}(x_4|\mu_2, 10) = 0.38 \times \mathcal{N}(35|11.32, 10) = 3.6542 \times 10^{-28}$$

$$\gamma(z_{52}) \propto \pi_2 \mathcal{N}(x_5|\mu_2, 10) = 0.38 \times \mathcal{N}(40|11.32, 10) = 3.0394 \times 10^{-36}$$