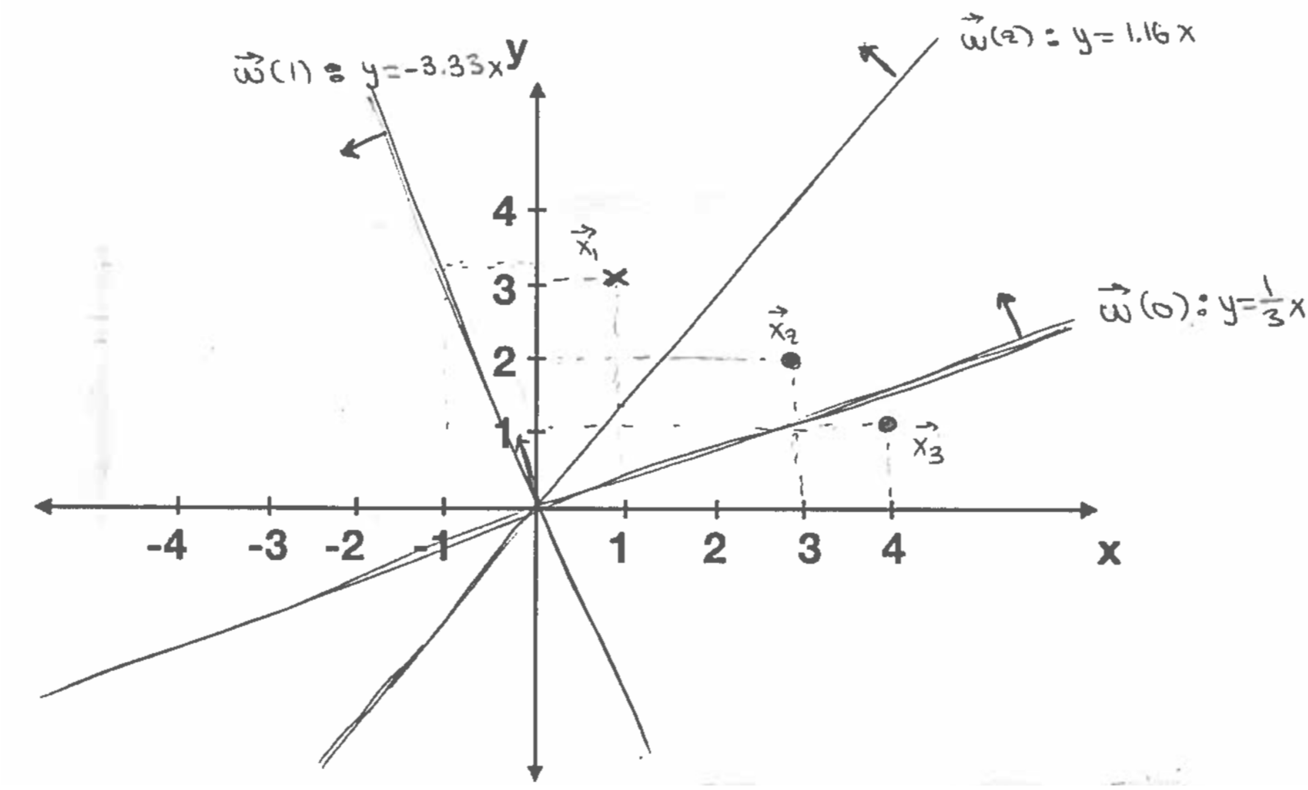
The goal of this problem is to run a linear perceptron algorithm. Assume that you have three training samples:

1. Sample $\mathbf{x_1} = [1, 3]^T$ from Class 1 ($y_1 = 1$)
2. Sample $\mathbf{x_2} = [3, 2]^T$ from Class 2 ($y_2 = -1$)
3. Sample $\mathbf{x_3} = [4, 1]^T$ from Class 2 ($y_3 = -1$)

The linear perceptron is initialized with a line with corresponding weight $\mathbf{w(0)} = [-\frac{1}{3}, 1]^T$. In the following, for the sake of simplicity, you will assume that all lines of the perceptron intersect point $(0, 0)$, therefore **you do not have to include any intercept w_0 or x_0 in the following calculations.**



(1) Plot $\mathbf{x_1}$, $\mathbf{x_2}$, and $\mathbf{x_3}$ in the given 2D space. Find and plot the line corresponding to weight $\mathbf{w(0)}$.

For $\mathbf{w(0)} = \left[-\frac{1}{3}, 1\right]$, we have $-\frac{1}{3}x + y = 0 \Rightarrow y = \frac{1}{3}x$, therefore the line is plot in the above figure. Note that the direction of the vector $\mathbf{w_0}$ on the line is the same as the direction of $\mathbf{w(0)}$ starting from $(0, 0)$.

(2) Using the rule $sign(\mathbf{w(t)}^T\mathbf{x_n})$, please indicate in which class are samples $\mathbf{x_1}$, $\mathbf{x_2}$, and $\mathbf{x_3}$ classified using the weight $\mathbf{w(0)}$. Which samples are not correctly classified based on this rule? **Note:** You have to compute the inner product $\mathbf{w(0)}^T\mathbf{x_n}$, $n = 1, 2, 3$, and see if it is greater or less than 0.

$\mathbf{w(0)}^T\mathbf{x_1} = -\frac{1}{3} + 3 = \frac{8}{3} > 0$, therefore sample $\mathbf{x_1}$ is correctly classified using $\mathbf{w(0)}$.

$\mathbf{w(0)}^T\mathbf{x_2} = -1 + 2 = 1 > 0$, therefore sample $\mathbf{x_2}$ is incorrectly classified using $\mathbf{w(0)}$.

$\mathbf{w(0)}^T\mathbf{x_3} = -\frac{4}{3} + 1 = -\frac{1}{3} < 0$, therefore sample $\mathbf{x_3}$ is correctly classified using $\mathbf{w(0)}$.

(3) Using the weight update rule from the linear perceptron algorithm, please find the value of the new weight $\mathbf{w(1)}$. Find and plot the new line corresponding to weight $\mathbf{w(1)}$ in the 2D space.
Note: The update rule is $\mathbf{w(t+1)} = \mathbf{w(t)} + y_s\mathbf{x_s}$, where $\mathbf{x_s}$ and $y_s \in \{-1, 1\}$ is the feature and class label of missclassified sample s.

$\mathbf{w(1)} = \mathbf{w(0)} - \mathbf{x_2} = \left[-\frac{1}{3}, 1\right]^T - [3, 2]^T = \left[-\frac{10}{3}, -1\right]^T$.

Note the "-" sign in the above equation, which is because the misclassified sample from $\mathbf{w(0)}$, $\mathbf{x_2}$, belongs to Class 2 ($y_2 = -1$).

The corresponding line is $-\frac{10}{3}x - y = 0 \Rightarrow y = -\frac{10}{3}x \Rightarrow y = -3.33x$

(4) Using the rule $sign(\mathbf{w(t)}^T\mathbf{x_n})$, please indicate in which class are samples $\mathbf{x_1}$, $\mathbf{x_2}$, and $\mathbf{x_3}$ classified using the weight $\mathbf{w(1)}$. Which samples are not correctly classified based on this rule?
Note: You have to compute the inner product $\mathbf{w(1)}^T\mathbf{x_n}$, $n = 1, 2, 3$, and see if it is greater or less than 0.

$\mathbf{w(1)}^T\mathbf{x_1} = -\frac{10}{3} - 3 = -\frac{19}{3} < 0$, therefore sample $\mathbf{x_1}$ is incorrectly classified using $\mathbf{w(1)}$.

$\mathbf{w(1)}^T\mathbf{x_2} = -10 - 2 = -12 < 0$, therefore sample $\mathbf{x_2}$ is correctly classified using $\mathbf{w(1)}$.

$\mathbf{w(1)}^T\mathbf{x_3} = -\frac{40}{3} - 1 = -\frac{43}{3} < 0$, therefore sample $\mathbf{x_3}$ is correctly classified using $\mathbf{w(1)}$.

(5) Using the weight update rule from the linear perceptron algorithm, please find the value of the new weight $\mathbf{w(2)}$. Find and plot the new line corresponding to weight $\mathbf{w(2)}$ in the 2D space. How many samples are correctly classified now?
Note: The update rule is $\mathbf{w(t+1)} = \mathbf{w(t)} + y_s\mathbf{x_s}$, where $\mathbf{x_s}$ and $y_s \in \{-1, 1\}$ is the feature and class label of missclassified sample s.

$\mathbf{w(2)} = \mathbf{w(1)} + \mathbf{x_1} = \left[-\frac{10}{3}, -1\right]^T + [1, 3]^T = \left[-\frac{7}{3}, 2\right]^T$.

Note the "+" sign in the above equation, which is because the misclassified sample from $\mathbf{w(1)}$, $\mathbf{x_1}$, belongs to Class 1 ($y_1 = 1$).

The corresponding line is $-\frac{7}{3}x + 2y = 0 \Rightarrow y = \frac{7}{6}x \Rightarrow y = 1.16x$

$\mathbf{w(2)}^T\mathbf{x_1} = -\frac{7}{3} + 6 = \frac{11}{3} > 0$, therefore sample $\mathbf{x_1}$ is correctly classified using $\mathbf{w(2)}$.

$\mathbf{w(2)}^T\mathbf{x_2} = -7 + 4 = -3 < 0$, therefore sample $\mathbf{x_2}$ is correctly classified using $\mathbf{w(2)}$.

$\mathbf{w(2)}^T\mathbf{x_3} = -\frac{28}{3} + 2 = -\frac{22}{3} < 0$, therefore sample $\mathbf{x_3}$ is correctly classified using $\mathbf{w(2)}$.

All samples are classified correctly based on $\mathbf{w(2)}$.